Stochastic Modelling and Applications
Volume - 6, No. 1, 2003

A Simple Estimate of the Index of Stability for Symmetric Stable Distributions

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Abstract

We propose an estimator of the stability parameter α of a stable distribution through the idea of ``wrapping." The advantage of this method, apart from being simple to calculate, is that the estimator can be shown to be consistent and asymptotically normal. The asymptotic variance is also easy to calculate so that one can actually provide confidence intervals and carry out inference on α for modestly large samples. The performance of the estimator, the confidence intervals and coverage probabilities are studied using simulated data and provide highly satisfactory results, compared to many existing procedures.

Keywords and Phrases: *symmetric stable, wrapped models, method of moments, asymptotic normality, confidence intervals, coverage probability*

AMS 1991 subject Classifications : Primary 62F10, 60E07, Secondary 62F12, 62F25

1. Introduction

Sample data arising in a wide variety of applications like economics (Mandelbrot, 1963), telephony (Stuck and Kleiner, 1974), radar clutter modelling (Jakeman and Pusey, 1976), and environmental sciences (Kogan and Manolakis, 1996) show large variability. In addition the histogram also exhibits a sharp peak. The family of stable distributions have been shown to be very useful and appropriate in modeling such heavy tailed distributions (Feller, 1971, Zolotarev, 1986, Samorodnitsky and Taqqu, 1994).

Stable distributions are an attractive option for modeling for two reasons. The first being that the sum of i.i.d. stable random variables will retain the shape of the original distribution, which is known as the stability property (Feller 1971). For example, one would like the distribution of the sum of daily changes in the stock price over a week to match the distribution of the weekly change. Secondly the stable distributions constitute a domain of attraction for sums of independent random variables. That is, if the sum of independently and identically distributed (i.i.d.) random variables converges in distribution, then the limiting distribution belongs to the stable family. This is desirable for modeling a phenomenon that is a superposition of a large number of random events. However, the stable distribution suffers from two major drawbacks. It lacks a simple closed form expression for the probability density function except in a few cases like the Gaussian (with $\alpha = 2$), Cauchy (with $\alpha = 1$) and Levy distributions. The second problem in using stable distributions with index of stability $\alpha \in (0,2)$ is that it possesses absolute moments only of order $p < \alpha$. In particular, the variance of these random variables is infinite except for the case of the normal distribution. This leads to a serious problem in estimation, since in the absence of nice asymptotic properties, providing confidence intervals or carrying out inference becomes very difficult.

The parameter that is of most interest in applications is the stability index α which determines how ``heavy" the tail is. Various estimators have been proposed for estimation of $\alpha$ which generally fall in three categories: maximum likelihood, quantile methods and characteristic function based methods. There are many estimators based around the so-called ``Hill estimator." Pictet, Dacorogna and Muller (1998) carry out a detailed simulation study of the performance of four such estimators, the Pickands estimator (Pickands, 1975, Dekkers and De Haan, 1989), the Hill estimator, De Haan and Resnick estimator (De Haan and Resnick, 1980) and a modified Hill estimator (Dekkers et. al., 1990). These ``Hill-type estimators" are based on the k largest observations in the sample. These estimators have nice asymptotic properties and are based on sound theory, but are not very satisfactory in practice (Resnick, 1997, 1998, Pictet, Dacorogna and Muller, 1998). These estimators vary considerably with the choice of k. Hall and Welsh (1985) derive an optimal k that depends on some parameters of the unknown distribution. Pictet, Dacorogna and Muller (1998) discuss estimation of these parameters and the performance of the resultant estimator. Drees and Kaufman (1998) propose a sequential approach to this problem and show that their estimator is asymptotically as efficient as the Hill estimator based on the optimal k. Drees (2001) examines asymptotic minimax risk bounds under zero-one loss and their implications for confidence intervals.

DuMouchel (1973) proposed a maximum likelihood type algorithm, which although theoretically superior, is computationally very expensive. A program to compute the maximum likelihood estimate is available online at http://academic2.american.edu/~jpnolan/. This program needs to be initialized, for which the simple estimator proposed in this paper can be a natural candidate.

Quantile methods are based on order statistics and are computationally inexpensive. Fama and Roll (1971) proposed the first quantile method which was improved later by McCulloch (1986). McCulloch's estimator works for $\alpha \in [0.6,2.0]$ (Adler, Feldman and Gallagher, 1998). Of the estimation techniques that use characteristic functions, the method by Koutrouvelis (1981) was shown to have the best performance. An improved version that greatly reduces the computation was presented by Kogon and Williams (1998). The McCulloch's estimator based on matching the quantiles and the Koutrouvelis estimator based on regression using the sample characteristic function give the best performance. For the McCulloch's estimator, the lack of asymptotics and the need to carry all the quantile tables makes it a bit cumbersome. For the Koutrouvelis estimator the choice of points where the regression sample characterisic function is estimated is based on a look-up table. Koutrouvelis and Baur (1982) have shown consistency and asymptotic normality for this estimator when these regression points are fixed. In practice, these points are determined based on the sample size and the value of α. Recently, Deo (2000) has provided estimators of the tail index based on U-statistics.

Asymptotic normality is proved for these estimators and are shown to have uniformly better efficiency than the regression based estimator of Koutrouvelis (1980). Deo (2000) also provides a goodness of fit test based on his estimator.

We propose an estimator based on the idea of ``wrapping." It is a trigonometric method of moments estimator derived using the simple properties of circular distributions. The proposed estimator is scale invariant. The performance of this estimator as compared with that of the Hill estimator is marginally better for α close to 1 but much superior for α close to 2 (compare Kogan and Williams (1998), Figure 4a and Table III of this paper). Note that an optimal choice of k for minimizing the MSE of the Hill estimator would result in a larger bias (Pictet, Dacorogna and Muller, 1998). However, the performance of our estimator is worse than that of the McCulloch's or the Koutrouvelis estimators. The advantage of the proposed estimator, apart from being simple to calculate, is that it is consistent and asymptotically normal. The asymptotic variance is also easy to calculate so that one can actually provide confidence intervals and carry out inference on α for moderately large samples. Inference can also be made using this estimator on whether the sample is from a stable population.

1.1 Stable Distributions

We denote an α – stable random variable by $S_\alpha(\sigma,\beta,\mu)$, where $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma \in \Re_+$, and $\mu \in \Re$ are the indexes of stability, skewness, scale and shift, respectively. When $\beta = 0$, the sub-family $S_\alpha(\sigma,0,\mu)$ is symmetric about μ. The problem with the stable family is that the density does not admit a closed form expression except in two cases. These are the well known cases of $S_2(\sigma,0,\mu)$ or the normal family with mean μ and variance $2N^2$ and the Cauchy family S_1 (N,0,:) with density $2\sigma/[\pi((x-\mu)^2+4\sigma^2)]$. The problem of not having a closed form of the density is somewhat mitigated by the presence of a closed form for the characteristic function $\phi(t)$ given by

$$\phi(t) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha [1 - i\beta \operatorname{sgn}(t) \tan\frac{\alpha\pi}{2}] + i\mu t\}, & \text{if } \alpha \in (0,1) \cup (1,2], \\ \exp\{-\sigma |t| [1 + \frac{2i\beta}{\pi} (sign\, t) \ln|t|] + i\mu t\}, & \text{if } \alpha = 1. \end{cases} \tag{1.1}$$

1.2 Wrapped Stable Distributions

The curse of infinite variance of a stable random variable X can be overcome, without losing the information on the tail index α by the following simple device of wrapping. Corresponding to any linear random variable X, say with density g and distribution function G, one may define a circular random variable $\theta = X \bmod (2\pi)$. Such circular random variables $0 \le \theta < 2\pi$ represent directions in two-dimensions and there is considerable theory behind these. See Jammalamadaka and SenGupta (2001, Sections 2.2.6 - 2.2.8), for more information on wrapped circular models. The density f and the distribution function F of such a wrapped variable, θ are obtained by the usual transformation (many-to-one in this case) theory, giving

$$f(u) = \sum_{k=-\infty}^{\infty} g(u + 2k\pi), \qquad \text{for } u \in [0,2\pi), \tag{1.2}$$

and

$$F(u) = \sum_{k=-\infty}^{\infty} [G(u + 2k\pi) - G(2k\pi)], \qquad \text{for } u \in [0,2\pi). \tag{1.3}$$

The density function of such a wrapped α – stable random variable for $\theta \in [0,2\pi)$, is given by

$$f(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{\infty} \exp\{-\sigma^\alpha j^\alpha\} \cos\{j(\theta - \mu) - \sigma^\alpha j^\alpha \beta \tan \frac{\alpha\pi}{2}\}, \tag{1.4}$$

when $\alpha \in (0,1) \cup (1,2]$, and μ can be conveniently redefined as $\mu := \mu \bmod (2\pi)$. Sometimes, σ is reparametrized by using the concentration parameter $\rho = \exp\{-\sigma^\alpha\}$ (see Jammalamadaka and SenGupta, 2001 equation (2.2.18)).

The main results that we need to know about such wrapped variables θ are (i) the characteristic function for such periodic variables need only be calculated at integer points (and these correspond to the Fourier coefficients in the Fourier series expansion of the density function, which is periodic) and (ii) the value of its characteristic function at these integer values is identical to that of X (see Jammalamadaka and SenGupta, 2001 Proposition 2.1).

In this paper we consider the problem of estimating the stability index for the *symmetric* stable family $S_\alpha(\sigma, 0, \mu)$ with $\beta = 0$. Then from (1.1), the corresponding wrapped variable in the class $S_\alpha(\sigma, 0, \mu)$ has the characteristic function (or Fourier coefficients)

$$E[e^{i(\theta-\mu)t}] = E[e^{i(X-\mu)t}] = e^{-\sigma^\alpha |t|^\alpha}, \tag{1.5}$$

where t is an integer. Estimation of σ and inference for wrapped symmetric α – stable circular models is studied in Gatto and Jammalamadaka (2001) where the stability index α is assumed to be known. On the other hand, the focus here is the estimation of α and it is indeed fortuitous that our estimator given in the next section, is independent of the two nuisance parameters μ and σ.

1.3 Estimation of α

We now propose what could be called the "trigonometric method of moment estimator" for α. Recall that if we define

$$\rho = e^{-\sigma^\alpha},$$

from (1.5),

$$E[e^{i(\theta-\mu)}] = E[\cos(\theta - \mu) + i \sin(\theta - \mu)] = \rho,$$

which implies that

$$\mu_{C_1} = E[\cos(\theta - \mu)] = \rho = \exp\{-\sigma^\alpha\} \text{ and } \mu_{S_1}(\theta - \mu)] = 0. \tag{1.6}$$

Similarly, taking $t = 2$ in (1.5), we conclude that

$$\mu_{C} = E[\cos(2(\theta-\mu))] = \rho^{2} = \exp\{-\sigma \;.2\;\} \text{ and } \mu_{S_2} = E[\sin(2(\theta-\mu))] = 0. \quad (1.7)$$

The method of (trigonometric) moments estimation involves equating these theoretical moments in equations (1.6) and (1.7) to the corresponding sample moments and solving for the parameters of interest. The sample moments are obtained as follows: Corresponding to i.i.d. observations $X_1, X_2, \ldots, X_n$ from a $S_\alpha(\sigma, 0, \mu)$ family, define the wrapped stable random variables

$$\theta_i = X_i \bmod 2\pi = \begin{cases} X \bmod 2\pi & \text{if } X \ge 0 \\ 2\pi - (|X| \bmod 2\pi) & \text{if } X < 0 \end{cases} \quad (1.8)$$

$i = 1, 2, \ldots, n$. Define the sequences of statistics

$$C_{1n} = \frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i) \qquad C_{2n} = \frac{1}{n}\sum_{i=1}^{n}\cos(2\theta_i)$$

$$S_{1n} = \frac{1}{n}\sum_{i=1}^{n}\sin(\theta_i) \qquad S_{2n} = \frac{1}{n}\sum_{i=1}^{n}\sin(2\theta_i) \quad (1.9)$$

The moment estimator of the unknown mean direction μ is given by the quadrant-specific $\tan^{-1}$,

$$\hat{\mu} = \arctan(S_{1n}/C_{1n}) = \bar{\theta}_0, \text{ say.}$$

Then it is simple to verify that

$$R_{1n} := \frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i - \bar{\theta}_0) = \sqrt{C_{1n}^2 + S_{1n}^2} \quad (1.10)$$

$$R_{2n} := \frac{1}{n}\sum_{i=1}^{n}\cos 2(\theta_i - \bar{\theta}_0) = \sqrt{C_{1n}^2 + S_{2n}^2} \quad (1.11)$$

are the corresponding sample central trigonometric moments of order 1 and order 2. Therefore, equating (1.6) and (1.7) to (1.10) and (1.11) respectively and solving for α, we get an estimate for α.

Definition: The estimate of α based on wrapped stable random variables is given by

$$\hat{\alpha} = \frac{1}{\ln 2}\ln\left(\frac{\ln R_{2n}}{\ln R_{1n}}\right). \quad (1.12)$$

Equivalently,

$$\hat{\alpha} = g(C_{1n}, S_{1n}, C_{2n}, S_{2n}) = \frac{1}{\ln 2}\left(\frac{\ln(C_{2n}^2 + S_{2n}^2)}{\ln(C_{1n}^2 + S_{1n}^2)}\right). \quad (1.13)$$

Remark 1. In the special case when μ in the model is assumed known, a new ``centered" X^* can be defined to be $(X-\mu)$ and the corresponding new θ^* as $(X^* \bmod 2\pi)$. In this case, the estimated mean direction $\bar{\theta}_0$ is replaced by the known mean direction μ in (1.10) and (1.11). The estimation of α proceeds exactly as in (1.12). This is analogous to the case of estimating the variance in a normal distribution in the case of a known mean.

One would expect to see some gain in the estimation of α. Unfortunately our simulations indicate such a gain was minimal at best and in any case it is not such a practical case.

2. Asymptotic Properties of the Estimator

Since in (1.9), we are dealing with the sums of i.i.d. random variables with finite variance, by the strong law of large numbers we know that $C_{1n} \to \mu_{C_1}$, $C_{2n} \to \mu_{C_2}$, and $S_{1n}, S_{2n} \to \mu_{S_1} = \mu_{S_2} = 0$ almost surely as $n \to \infty$. Strong consistency of the estimator $g(C_{1n}, S_{1n}, C_{2n}, S_{2n})$ to α follows by noting that g is continuous and $g(\mu) = \alpha$ where $\mu = (\mu_{C_1}, \mu_{S_1}, \mu_{C_2}, \mu_{S_2})$.

Further, an application of the multivariate central limit theorem yields

$$\sqrt{n}((C_1, S_1, C_2, S_2)' - \mu') \xrightarrow{d} N(0, \Sigma), \qquad n \to \infty \tag{2.1}$$

where Σ is the matrix of covariances and $\xrightarrow{d}$ denotes convergence in distribution. Let $c = (\ln 2)^{-1}$.

Theorem 2.1 As $n \to \infty$, $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, \sigma^2)$, where

$$\sigma^2 = c^2\left(\frac{1}{\rho^2}\sigma_{C_1}^2 + \frac{1}{\rho^{2(\alpha+1)}2^{2\alpha}}\sigma_{C_2}^2 - \frac{1}{\rho^{(2^\alpha+1)}2^\alpha}\sigma_{C_1,C_2}\right), \tag{2.2}$$

with

$$\sigma_{C_1}^2 = \mathrm{Var}(\cos(\theta)) = \frac{1+\rho^{2^\alpha}}{2} - \rho^2, \tag{2.3}$$

$$\sigma_{C_2}^2 = \mathrm{Var}(\cos(2\theta)) = \frac{1+\rho^{4^\alpha}}{2} - \rho^{2^{(\alpha+1)}}, \tag{2.4}$$

and

$$\sigma_{C_1,C_2} = \mathrm{Cov}(C_1, C_2) = \frac{\rho^{3^\alpha} + \rho}{2} - \rho^{(2^\alpha+1)}. \tag{2.5}$$

Proof. Using the delta method (see Rao (1985, pp387, (6a.2.6)), we get

$$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} \mathrm{N}(0, \sigma^2),$$

as $n \to \infty$ where

$$\sigma^2 = \mathrm{Var}(\hat{\alpha}) = \mathrm{g}_{C_1}^2(\mu)\sigma_{C_1}^2 + \mathrm{g}_{C_2}^2(\mu)\sigma_{C_2}^2 + 2\mathrm{g}_{C_1}(\mu)\mathrm{g}_{C_2}(\mu)\sigma_{C_1,C_2}, \tag{2.6}$$

where $g_x(x, y)$ denotes the derivative of g w.r.t. x etc. Note that the terms in S_1 and S_2 do not appear in (2.6) due to the fact that $g_{S_1}(\mu) = g_{S_2}(\mu) = 0$. (2.2) now follows from (2.6) by evaluating the derivatives of g.

Calculating variances of C_1 and C_2 is quite easy:

$$E[(e^{i\theta})^2] = E(\cos(\theta) + i\sin(\theta))^2 = E(\cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta)).$$

Also $E[(e^{i\theta})^2] = E[e^{2i\theta}] = \rho^{2^\alpha}$.

Equating real and imaginary parts we get $E(\cos^2(\theta) - \sin^2(\theta)) = \rho^{2^\alpha}$.

This together with $E(\cos^2(\theta) + \sin^2(\theta)) = 1$ gives

$$E[\cos^2(\theta)] = \frac{1+\rho^{2^\alpha}}{2}.$$

This together with (1.6) gives (2.3). $\sigma^2_{C_2}$ can be calculated similarly.

We now evaluate $\mathrm{Cov}(C_1, C_2)$.

$$E[e^{i\theta}e^{2i\theta}] = \rho^{3^\alpha} \tag{2.7}$$

Expanding the L.H.S gives us

$$E[\cos\theta\cos 2\theta - \sin\theta\sin 2\theta] = \rho^{3^\alpha} \tag{2.8}$$

Similarly,

$$E[\cos\theta\cos 2\theta + \sin\theta\sin 2\theta] = \rho \tag{2.9}$$

Adding (2.8) and (2.9) gives

$$E[\cos\theta\cos 2\theta] = \frac{\rho^{3^\alpha}+\rho}{2} \tag{2.10}$$

Thus,

$$\mathrm{Cov}(C_1, C_2) = \frac{\rho^{3^\alpha}+\rho}{2} - \rho^{(2^\alpha+1)} \tag{2.11}$$

This proves Theorem 2.1

A straightforward but important corollary of the above theorem is the following:

Corollary 2.2 An approximate $100(1-\gamma)\%$ confidence interval for α for a large enough sample of size n is given by

$$[\max(\hat{\alpha} - \frac{z_{\gamma/2}\sigma}{\sqrt{n}}, 0), \min(\hat{\alpha} + \frac{z_{\gamma/2}\sigma}{\sqrt{n}}, 2)], \tag{2.12}$$

where $z_{\gamma/2}$ is such that $P(Z > z_{\gamma/2}) = \gamma/2$, and Z is a standard normal random variable.

3. Computational Results

Simulation of stable random variables is done using the Chambers, Mallows and Stuck method (see Samordnisky and Taqqu (1994)). The location parameter μ is taken to be zero and the scale parameter σ is taken to be unity. There are two parts to this section on computational results. Table I give the theoretical asymptotic variances σ^2 and length of the confidence intervals for various values of α for sample sizes $n = 1000$ and $n = 10{,}000$ Next we simulate 1000 samples with the above two choices of sample size. Table II list the actual value of α, the average of the estimate $\hat{\alpha}$ and mean square errors over 1000 samples and the average length of the 95% confidence intervals. Table III compares the average and MSE of the McCulloch's estimator and the estimator using wrapped stable proposed in this paper.

In the above simulation, for the cases $\alpha = 0.2$, and 0.3, out of the 1000 samples there were 18 and 1 samples respectively for which the estimate of alpha turned out to be negative. These samples were dropped from the study. Likewise, for the values of α ranging from 1.4 to 1.9 the number of samples for which the estimate was found to be greater than 2 (and hence was taken to be equal to 2), were 4, 8, 40, 58, 108, and 137 respectively.

With samples of size 10,000 there were no negative estimates as in the previous case. For α values 1.7, 1.8, and 1.9 out of the 10,000 samples 1, 24, and 159 samples had values of estimate greater than 2. Another point to be noted is that for extreme values of α, the Confidence Interval (CI) may not be symmetric around $\hat{\alpha}$.

4. Comparison with other Estimators

We compare the average of the estimates of α and the corresponding MSE of the McCulloch's estimator with the above estimator (see Table III). The size of the sample is taken to be 1000 and we replicate 10,000 times. The data for the McCulloch's estimator which is a quantile based method is taken from Adler, Feldman and Gallagher (1998). Since the number of replications (10,000) is fairly large, we feel that the fact that the samples are different do not really matter. McCulloch's estimator has a performance comparable with methods based on the characterestic function. McCulloch's estimator has a slightly lower MSE for smaller values of α as compared to the characterestic function based estimators while the roles are reversed for values of α close to 2 (Kogan and Williams, 1998). Further, these methods are much superior in performance to methods based on extreme value theory, for symmetric stable distributions.

The most attractive feature of the proposed estimator is the ease of computation and independence from tables as in the case of the McCulloch's estimator. The McCulloch's estimator has a lower MSE than the estimator proposed in this paper. The estimator based on wrapped stable distribution performs better marginally than the method based on extreme value theory (Hill-type estimator) for $\alpha = 1.0$ and 1.2, while its performance is much superior for α close to 2 (compare Kogan and Williams (1998), Figure 4a with Table III). Further, for the Hill estimator, the optimal choice of the number of largest order statistics to be used (k) is a compromise between the bias and the MSE. Choosing k to minimise MSE will result in larger bias and vice-versa (Pictet, Dacorogna and Muller (1998)).

It must be mentioned here that in the absence of any asymptotic results or the estimator having infinite variance, the MSE does not have much use. With the estimator proposed in this paper, we can construct confidence intervals and carry out inference.

An important application of the asymptotic normality of our estimator is that it can be used to test if a sample is from the stable family. This can be done as follows: Divide the data set into three parts, the last two parts being of equal size. Let n_1 and n_2 be the sample sizes in the first and second parts. Let $\hat{\alpha}_1$ be the estimate for α from the first part.

Next we estimate α from the data $(x_{2i} + x_{3i})/2$, where x_{ki} is the ith sample observation from the kth part. Call this estimate $\hat{\alpha}_2$. If the original data is from a symmetric stable, then the two estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$ should be close. This test can be carried out using the asymptotic result of Theorem 2.1. A study of the efficacy of this procedure is being carried out.

Table I
Asymptotic variance and length of 95% CI for sample sizes n=1000 and n=10,000

α	Asymptotic Variance		Length of 95% CI	
	n=1000	n=10,000	n=1000	n=10,000
0.2	0.009	0.0009	0.389	0.123
0.3	0.009	0.0009	0.391	0.123
0.4	0.010	0.0010	0.395	0.124
0.5	0.010	0.0010	0.399	0.126
0.6	0.010	0.0010	0.406	0.128
0.7	0.011	0.0011	0.415	0.131
0.8	0.011	0.0011	0.427	0.135
0.9	0.012	0.0012	0.422	0.140
1.0	0.013	0.0013	0.462	0.146
1.1	0.015	0.0015	0.489	0.154
1.2	0.017	0.0017	0.523	0.165
1.3	0.020	0.0020	0.567	0.179
1.4	0.025	0.0025	0.625	0.197
1.5	0.031	0.0031	0.699	0.221
1.6	0.041	0.0041	0.797	0.252
1.7	0.055	0.0055	0.763	0.292
1.8	0.078	0.0078	0.748	0.346
1.9	0.113	0.0113	0.761	0.309

Table II
Coverage probabilities calculated using 1000 samples of sizes n=1000 and n=10,000 each

α	Median of $\hat{\alpha}$		Average of $\hat{\alpha}$		MSE		Coverage Prob.		Average Length of 95% CI	
	n=1000	n=10,000	n=1000	n=10,000	N=1000	n=10,000	n=1000	n=10,000	n=1000	n=10,000
0.2	0.201	0.199	0.210	0.199	0.0091	0.0010	0.966	0.950	0.358	0.123
0.3	0.294	0.301	0.295	0.300	0.0099	0.0010	0.949	0.955	0.384	0.123
0.4	0.392	0.399	0.392	0.398	0.0101	0.0011	0.944	0.938	0.394	0.124
0.5	0.501	0.501	0.500	0.501	0.0103	0.0011	0.956	0.953	0.400	0.126
0.6	0.599	0.598	0.597	0.598	0.0101	0.0011	0.970	0.944	0.407	0.128
0.7	0.694	0.700	0.695	0.609	0.0110	0.0011	0.954	0.954	0.416	0.131
0.8	0.800	0.799	0.797	0.797	0.0116	0.0011	0.958	0.955	0.429	0.135
0.9	0.896	0.900	0.898	0.899	0.0140	0.0013	0.942	0.938	0.445	0.140
1.0	0.995	0.997	1.001	0.997	0.0133	0.0015	0.966	0.950	0.467	0.146
1.1	1.091	1.099	1.093	1.098	0.0149	0.0015	0.962	0.956	0.493	0.154
1.2	1.195	1.196	1.198	1.197	0.0187	0.0020	0.950	0.936	0.532	0.165
1.3	1.285	1.299	1.289	1.300	0.0219	0.0020	0.953	0.958	0.573	0.179
1.4	1.374	1.397	1.384	1.399	0.0256	0.0027	0.948	0.949	0.622	0.198
1.5	1.468	1.499	1.483	1.500	0.0302	0.0031	0.938	0.952	0.672	0.222
1.6	1.567	1.594	1.582	1.599	0.0358	0.0042	0.943	0.954	0.714	0.254
1.7	1.633	1.693	1.647	1.697	0.0382	0.0053	0.926	0.958	0.734	0.292
1.8	1.698	1.789	1.707	1.794	0.0441	0.0076	0.887	0.950	0.750	0.316
1.9	1.733	1.891	1.743	1.885	0.0563	0.0078	0.866	0.942	0.760	0.305

Table III

Comparison of Mean and MSE (in parantheses) of 10,000 estimates of K using samples of size 1000. (Source: Adler, Feldman and Gallagher (1998))

Ş	McCulloch's Estimator	Estimator Based on Wrapping
0.6	0.633(0.00157)	0.601(0.01081)
0.8	0.804(0.00131)	0.799(0.01216)
1.0	1.002(0.00192)	0.998(0.01412)
1.6	1.606(0.00505)	1.576(0.03591)
1.8	1.808(0.00874)	1.710(0.04271)

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